UNIVERSAL ALGORITHM OF UNIFORM DISTRIBUTION OF POINTS ON ARBITRARY ANALITIC SURFACES IN THREE-DIMENSIONAL SPACE

N.P. Kopytov, E.A. Mityushov

Ural Federal University named after the first President of Russia B.N. Yeltsin, Ekaterinburg, Russia

Abstract

The problem of uniform distribution of points on arbitrary analytic surfaces in three-dimensional space is considered. A universal algorithm for uniform distribution of points on analytic surfaces defined by the parametric method is proposed. Neumann's method for generating a two-dimensional random variable by using a known density function of the joint distribution is described. Graphical presentations of the proposed algorithm obtained with the help of Wolfram Mathematica 7.0 are demonstrated. The examples of uniform distribution of points on surfaces of sphere, torus, helicoid, "falling drop" and surface of Klein bottle are presented.

Key words: uniform distribution of points on surfaces, acceptance-rejection method

1. Introduction

The problem of uniform distribution of points on surfaces is important for various field of science such as mathematical modeling, numerical modeling, apply mechanics, Monte Carlo techniques and many others.

There exists many papers and internet sources devoted to the problem of points uniform distribution on surface of sphere, see, for example, references [1-3] and references therein. Namely problem of uniform distribution of points on the spherical surface shows main difficulties associated with uniform distribution of surfaces. It is especially important to note the investigation by G. Melfi and G. Schoier, see in reference [4], where the algorithm of uniform distribution of points on surfaces $z = z(x, y)$ is presented. Their algorithm is closely related to our method.

The proposed in this paper algorithm is most general for uniform distribution of points on various surfaces because it allows to uniformly distribute points on arbitrary analytic surfaces in three-dimensional space which can be defined in the parametric form $\mathbf{r} = \mathbf{r}(u, v)$, that is by functions $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, defined on the domain $D = \{u_1 \le u \le u_2; v_1 \le v \le v_2\}$.

This definition allows us dial with such surfaces as torus, helicoid, Klein bottle and so on, and also it includes sphere surface and surfaces of explicit definition form such as surfaces $z = z(x, y)$.

The proposed algorithm consists in two main parts. The first one is the finding of density function of the joint distribution corresponding to uniform distribution of points on given surface. The second one is the generating of two-dimensional random variable using generalized Neumann's method called also acceptance-rejection method.

2. Statement of problem

Let parametric surface be defined by parametric functions

$$x = x(u,v),\ y = y(u,v),\ z = z(u,v),$$

where the parameters u and v are defined the domain $D = \{u_1 \le u \le u_2; v_1 \le v \le v_2\}$.

It is necessary to distribute uniformly points on this surface.

3. Density function finding

Let parametric surface be defined by parametric functions $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, where the parameters u and v are defined in the domain $D = \{u_1 \le u \le u_2; v_1 \le v \le v_2\}$.

It is necessary to find analytically a density function $f(u,v)$ of the joint distribution of parameters u and v corresponding to uniform distribution of points on the considered surface.

In the case when points are uniformly distributed on the considered surface, probability of entering of any point A on a surface element dS can be defined as

$$P(A \subset dS) = \frac{dS}{S},$$

where $dS = \sqrt{EG - F^2}dudv$, $S = \iint_D \sqrt{EG - F^2}dudv$, where E, F, G are the coefficients of the first fundamental form of surface, see [5]. And hence,

$$P(A \subset dS) = \frac{\sqrt{EG - F^2}dudv}{\iint_D \sqrt{EG - F^2}dudv}. \tag{3.1}$$

The probability a point A to appear on surface element dS can be also defined as

$$P(A \subset dS) = f(u,v)dudv, \tag{3.2}$$

where $f(u,v)$ is the required density function of the joint distribution of parameters u and v.

Taking into account expressions (3.1) and (3.2) we obtain

$$f(u,v) = \frac{\sqrt{EG - F^2}}{\iint\limits_D \sqrt{EG - F^2}\,dudv}. \tag{3.3}$$

By generating the values of u, v with the help of the obtained function $f(u,v)$, and then finding correspond coordinates of points, we obtain uniform distribution of points on the considered surface.

4. Generating multidimensional random variable by using a known density function of the joint distribution

Various methods are used to generate one-dimensional random variable by using a known density function of distribution, see [6]. For example, the inverse-transform method can be applied when the probability distribution function can be obtained analytically. However, application of this method meets difficulties in the cases of multidimensional distributions of dependent random variables. A universal method generation of one-dimensional random variable is the acceptance-rejection method known also as Neumann's method. Firstly, let us consider the acceptance-rejection method for the example of one-dimensional random variable generation by using function of the joint distribution. Then we consider a generalization of this method for multidimensional random variable generating by using a known density function of the joint distribution.

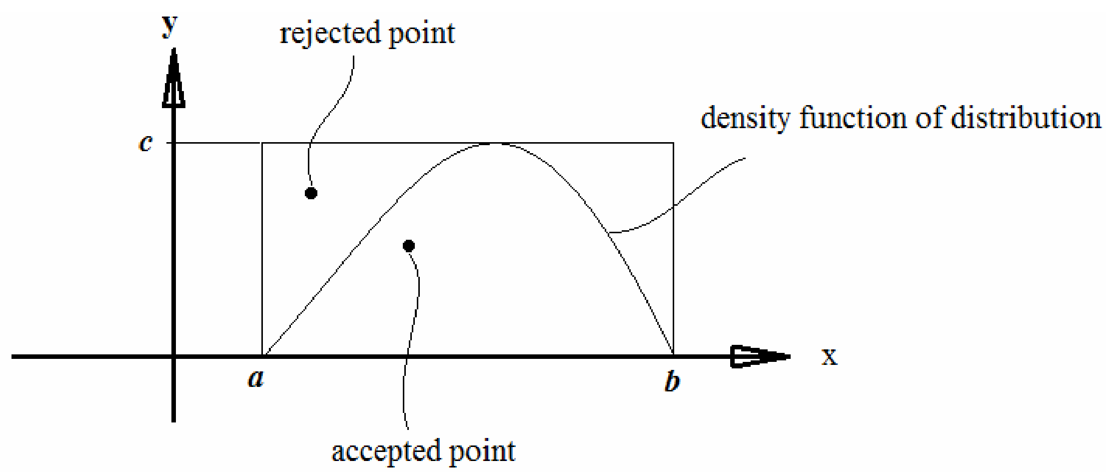

Fig. 4.1. This figure illustrates acceptance-rejection method for generation of one-dimensional random variable

In the case of generation of one-dimensional random variable, the acceptance-rejection method consists in the following steps (see fig. 4.1):

1) The density function of distribution is placed in the rectangle such a way as it is shown in figure 4.1;

2) One generates random point with coordinates $x = (b-a)\text{Random} + a$, $y = c\text{Random}$, where Random is random number on interval $(0,1)$;

3) The obtained point is accepted if it lies below the curve of density function of distribution. Otherwise, the point is rejected (see fig. 4.1);

4) One then repeats steps 2, 3.

In application to multi-dimensional cases the generation procedure is unchanged except one take into account changes in the dimensionality of space. For two-dimensional random variables corresponding to our case, obviousness of algorithm is unchanged, since accept-rejection procedure is carried out in the three-dimensional space. In this case, the algorithm is implemented as follows:

1) One finds $\max\limits_{D} f(u,v)$ - maximal value of function $f(u,v)$ in the domain $D = \{u_1 \le u \le u_2; v_1 \le v \le v_2\}$.

2) Two random numbers $u_0 = (u_2 - u_1)\text{Random}+u_1$, $v_0 = (v_2 - v_1)\text{Random}+v_1$ are generated, where Random is random number on interval $(0,1)$;

3) If $\text{Random}\times\max\limits_{D} f(u,v) < f(u_0,v_0)$, the point is accepted (here, Random is also random number on interval $(0,1)$). Otherwise, the point is rejected;

4) One repeats steps 2, 3.

5. Graphical presentation of results

The described above algorithms can be realized in the package Wolfram Mathematica 7.0, which allows to show visual models. It is very convenient to control performance of algorithm. Several results are presented below.

Example 1. A uniform distribution of 15000 points on a sphere are presented in figure 5.1. The surface of sphere is defined by following equations:

$$x = x(u,v) = \sin u \cos v, \quad y = y(u,v) = \sin u \sin v, \quad z = z(u,v) = \cos u,$$

where $0 \le u \le \pi$, $0 \le v \le 2\pi$.

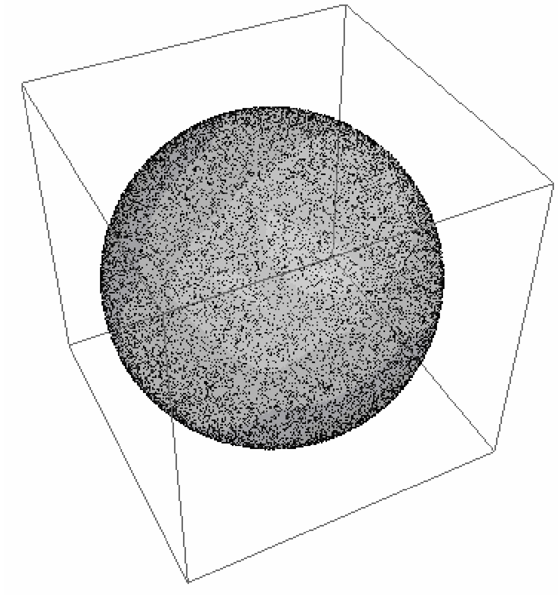

Fig. 5.1. A uniform distribution of 15000 points on the surface of sphere,
ViewPoint: {1,2,2}

Example 2. A uniform distribution of 20000 points on a torus is presented in figure 5.2. The surface of torus is defined by following equations:

$$x = x(u,v) = (3+\cos u)\cos v\,,\ \ y = y(u,v) = (3+\cos u)\sin v\,,\ \ z = z(u,v) = \sin u\,,$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

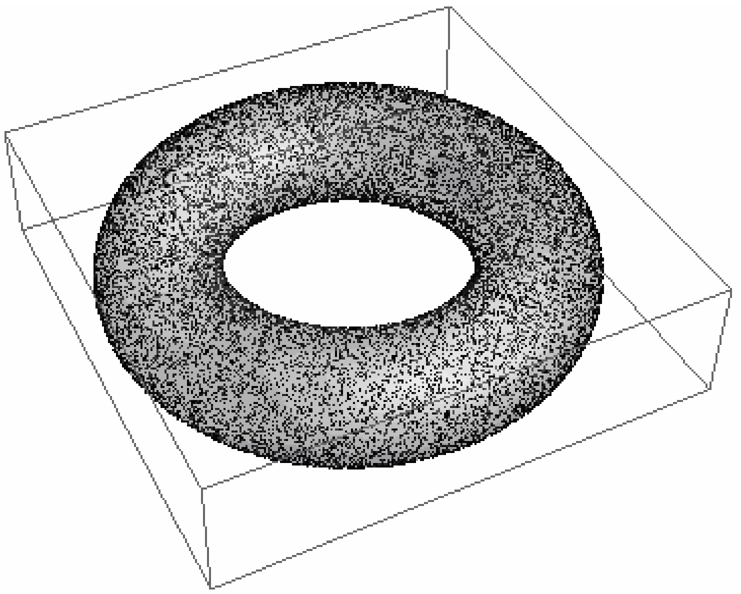

Fig. 5.2. A uniform distribution of 20000 points on the surface of torus,
ViewPoint: {1,2,2}

Example 3. A uniform distribution of 15000 points on a helicoid is presented in figure 5.3. The surface of helicoid is defined by the following equations:

$$x = x(u,v) = u\cos v\,,\ \ y = y(u,v) = u\sin v\,,\ \ z = z(u,v) = v\,,$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

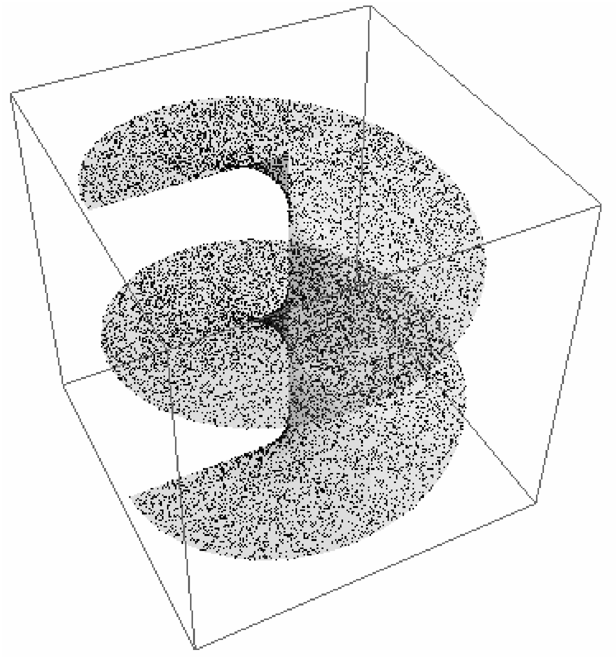

Fig. 5.3. A uniform distribution of 15000 points on the surface of helicoid,
ViewPoint: {1,2,2}

Example 4. A uniform distribution of 20000 points on the surface of "falling drop" is presented in figure 5.4. The surface of "falling drop" is defined by the following equations:

$$x = x(u,v) = v^2\sqrt{1-v}\cos u\,,\;\; y = y(u,v) = v^2\sqrt{1-v}\sin u\,,\;\; z = z(u,v) = -v\,,$$

where $0 \le u \le 2\pi$, $0 \le v \le 1$.

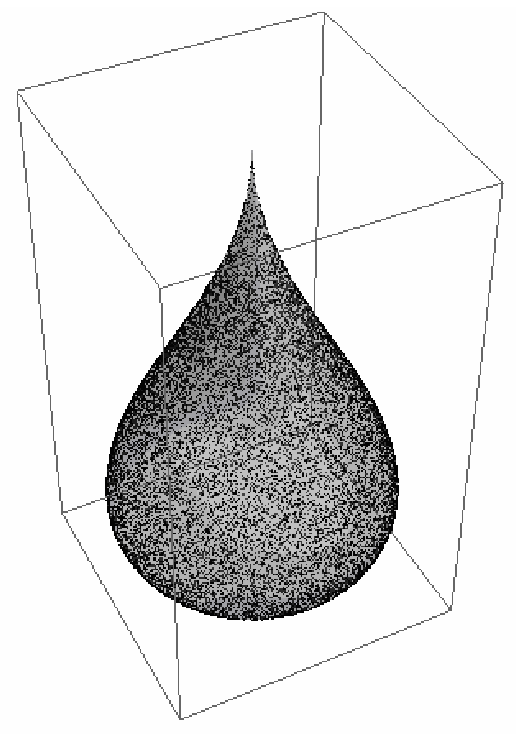

Fig. 5.4. A uniform distribution of 20000 points on the surface of "falling drop",
ViewPoint: {1,2,2}

Example 5. A uniform distribution of 15000 points on Klein bottle is presented in figure 5.5. Surface of Klein bottle is defined by the following equations [7]:

$$x = x(u,v) = -\frac{2}{15}\cos u(3\cos v + 5\sin u\cos v\cos u - 30\sin u - 60\sin u\cos^6 v + 90\sin u\cos^4 v),$$

$$y = y(u,v) = -\frac{1}{15}\sin u(80\cos v\cos^7 u\sin u + 48\cos v\cos^6 u - 80\cos v\cos^5 u\sin u -$$
$$-48\cos v\cos^4 u - 5\cos v\cos^3 u\sin u - 3\cos v\cos^2 u + 5\sin u\cos v\cos u + 3\cos v - 60\sin u),$$

$$z = z(u,v) = \frac{2}{15}\sin v(3 + 5\sin u\cos u),$$

where $-\pi/2 \le \pi/2$, $0 \le v \le 2\pi$.

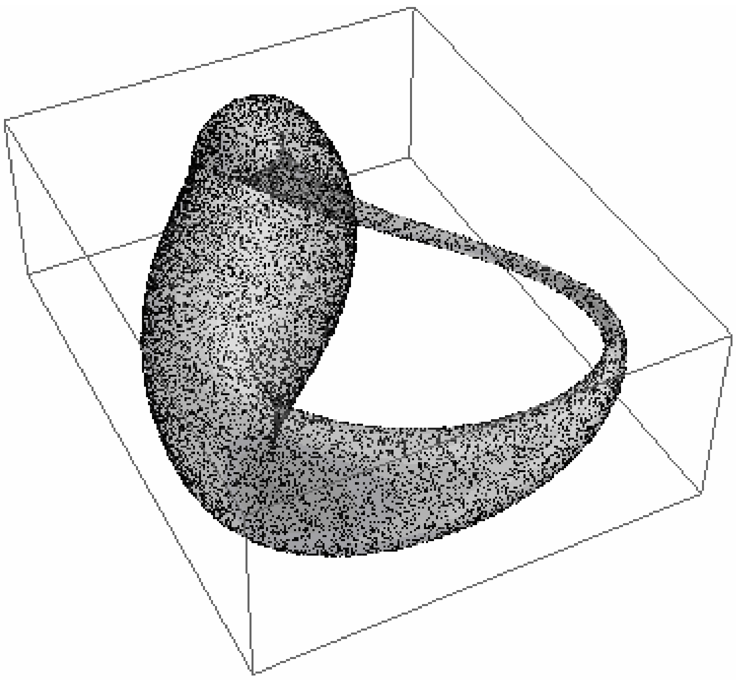

Fig. 5.5. Uniform distribution of 15000 points on surface of Klein bottle,
ViewPoint: {1,2,2}

6. Conclusions

The visual analysis of obtained results confirms efficiency of the proposed algorithm. The algorithm is applied to various surfaces. Therefore, this method may be useful as a tool for different investigations, especially, for research which use Monte Carlo techniques.

References

1. *Marsaglia G.* Choosing a Point from the Surface of a Sphere.
Ann. Math. Stat. 43, 645-646, 1972.

2. *Tashiro Y.* On methods for generating uniform random points on the surface of a sphere. Ann. Math. Stat. 43, 645-646, 1972.

3. *Weisstein, Eric W.* "Sphere Point Picking." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/SpherePointPicking.html

4. *Melfi G., Schoier G.* Simulation of random distributions on surfaces. http://www.sis-statistica.it/files/pdf/atti/RSBa2004p173-176.pdf

5. *Weisstein, Eric W.* "First Fundamental Form." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/FirstFundamentalForm.html

6. *Rubinstein R.Y., Kroese D.P.* Simulation and the Monte Carlo methods. Second Edition. Wiley-Intersciens, 2007. 345 p.

7. *Andrew J.P. Maclean*. Parametric Equations for Surfaces *https://wiki.sch.bme.hu/pub/Infoalap/SzgGraf/ParametricSurfaces.pdf*